# Passive Location Estimation Using TOA Measurements

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***Abstract*—Ultra-wideband (UWB) radios offer highly accurate ranging, i.e., measurement of signal travel time. This is important for positioning (localization), in particular in indoor environments, where GPS is not available. This paper considers UWB-based positioning of *passive* objects, where one transmitter and multiple, distributed, receivers are employed. We consider the case where the transmitter and receivers can be synchronized, so that time-of-arrival (TOA) instead of time-difference-of-arrival (TDOA) information can be utilized. Assuming Gaussian errors for the range estimates, we propose a novel, Two-Step, Expectation Maximization (TSEM) based algorithm for the localization of the passive object. This algorithm achieves the Cramer-Rao Lower Bound (CRLB) of TOA algorithms. Simulation results show that the error variance of TSEM is much lower (often 30 dB) than that of the existing, TDOA-based, algorithms.**

***Index Terms*—TOA, TDOA, location estimation, CRLB**

## I. Introduction

Positioning [1] techniques, which supply information of target location, are useful in many scenarios ranging from location-based services to security and emergency services [1]. The positioning problem was previously studied as "multistatic radar", and recently gained extensive attention for surveillance applications based on Ultra-wideband (UWB) technology. The UWB signaling is an ideal candidate for positioning [2] because of its high delay resolution and consequent accurate range (signal travel time) measurement. For these reasons, UWB is used for the IEEE 802.15.4a standard for ranging and communications in Wireless Sensor Networks (WSNs) [3], [4].

Because of their importance, various types of positioning techniques have been extensive studied in the literature. Based on the type of information and system parameters that are used, positioning techniques can be categorized into those that utilize (i) received signal strength (RSS) [5], (ii) angle of arrival (AOA), and (iii) signal propagation delay [2], [6], [7], respectively. RSS algorithms utilize the received signal power, but their accuracy is limited by the variations of channel attenuation [5]. AOA algorithms need antenna arrays to extract angle information. Signal propagation delay based algorithms estimate the object location based on the time it takes the signal to travel from the transmitter to receivers. They achieve very accurate estimation of object location if combined with high-precision timing measurement techniques [8]. For this reason, UWB signals are ideal candidates for propagation time based algorithms: their large bandwidth allows centimeter and even sub-millimeter accuracy [9] of the range measurements.

Positioning algorithms based on signal propagation time can be further classified into Time of Arrival (TOA) and Time Difference of Arrival (TDOA). TOA algorithms employ the information of the *absolute* signal travel time from the transmitter to the receivers; this approach requires the knowledge of signal departure time and thus the synchronization between the transmitter and receivers. Such synchronization can be done by cable connections between the devices, or sophisticated wireless synchronization algorithms [10]. TDOA is employed if there is no synchronization between the transmitter and the receivers. In that case, the receivers do not know the signal travel time and therefore employ the *difference* of signal travel times between the receivers. It is intuitive that TOA has better performance than the TDOA, since the TDOA loses information about the signal departure time.

The target in TDOA/TOA positioning can be an "active" or "passive" object. "Active" means that the object itself is the transmitter, while "passive" means that it just reflects signals stemming from a separate transmitter.

The active-object localization methods are used in mobile terminals, WLAN, satellite positioning, and RFID. Both TOA and TDOA methods for "active" objects have been studied extensively. For TDOA, the two-stage method [11] and the Approximate Maximum Likelihood Estimation [12] were shown to achieve the Cramer-Rao Lower Bound (CRLB) of active TDOA [6]. As is well-known, the CRLB sets the lower bound of the estimation error variance of any un-biased method. For TOA-based active-object positioning methods, which achieve the CRLB of active TOA, are the Least-Square Method [13] and the Approximate Maximum Likelihood Estimation Method [12].

Passive-object positioning is important in many practical situations like crime-prevention surveillance, assets tracking, and medical patient monitoring, where the target is a reflecting/scattering object that is unwilling/unable to carry a wireless transceiver. Passive TDOA positioning algorithms do not differ significantly from their *active* counterparts. For

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[1]as is common in much of the literature, we use the words "positioning" and "localization" interchangeably.

TOA, however, the synchronization creates a fundamental *difference* between "active" and "passive" cases; yet to the best of our knowledge, no passive-object TOA algorithms have been proposed.

This paper aims to fill this gap by proposing a novel, two step Expectation Maximization (TSEM) method for *passive* object location estimation, which achieves the CRLB of "passive" TOA positioning algorithms. It combines the Expectation Maximization technique and an idea from the TDOA algorithm of [11]. We show that the CRLB of TOA is much lower than that of TDOA. In other words, no unbiased TDOA or TOA method can outperform TSEM in terms of the estimation error variance.

The remainder of this paper is organized as follows. Section II presents the system model of object location estimation. Section III derives the TSEM. Section IV shows the simulation results. Finally Section V draws the conclusions.

Notation: Throughout this paper, a variable with "hat" $\hat{\bullet}$ denotes the measured/estimated values, and the "bar" $\bar{\bullet}$ denotes the mean value. Bold letters denote vectors/matrices. $E(\bullet)$ is the expectation operator. If not particularly specified, "TOA" in this paper denotes the "TOA" for passive object.

## II. System Model

For ease of exposition, we consider a two-dimensional passive object (target) location estimation problem as shown in Fig. 1. The positioning system consists of one transmitter and $M$ receivers. Without loss of generality, let the location of the transmitter be $[0,0]$, and the location of $i$th receiver be $[a_i, b_i]$, $1 \le i \le M$. The transmitter transmits an impulse; the receivers subsequently receive the signal reflected from the target, whose location $[x, y]$ is to be estimated, We adopt the assumption in [11], [12] that the target reflects (scatters) incident signals into all directions. Transmitter and receiver are assumed to be perfectly synchronized via wireless time transfer or through cable connections. Thus, at the estimation center, signal travel times can be obtained by comparing the departure time at the transmitter and the arrival time at the receivers.

Note that the total signal arriving at the receiver consists of reflections not only by the target, but also from other, non-target objects. However, by using background channel cancelation and advanced channel parameter extraction methods [14], the estimation center can eliminate these unwanted signal components and extract the reflection from the target object.

Let the signal travel time from the transmitter to the $i$th receiver be $t_i$, and $r_i = c_0 t_i$, where $c_0$ is the speed of light, $1 \le i \le M$. Then,

$$r_i = \sqrt{x^2+y^2} + \sqrt{(x-a_i)^2+(y-b_i)^2} \quad i = 1, ...M. \quad (1)$$

For future use we define $\mathbf{r} = [r_1, r_2, \dots, r_M]$. If the measurements are error-free, the location estimate of the target $[\hat{x}, \hat{y}]$ is identical to the true location $[x, y]$. However, measurement errors inevitably exists, so that

$$r_i = \hat{r}_i + e_i, \qquad 1 \le i \le M,$$

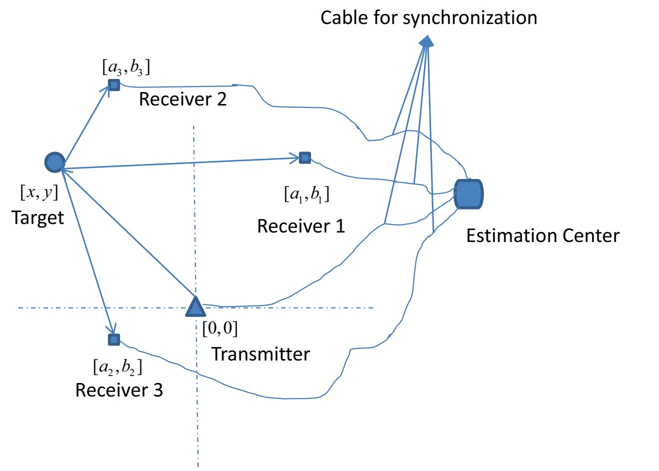



Figure 1. Illustration of TOA based Location Estimation System Model

where $r_i$ is the true value, $\hat{r}_i$ is the measured value and $e_i$ is the measurement error. This paper does not address the topic of TOA error characteristics, and please refer to [15] for more detailed information. The estimation algorithm aims to find the $[\hat{x}, \hat{y}]$ that best fits the above equations in the sense of minimizing the error variance

$$\Delta = E[(\hat{x}-x)^2 + (\hat{y}-y)^2]. \quad (2)$$

Assuming that the $e_i$ are Gaussian-distributed variables with variances $\sigma_i^2$, the conditional probability function of the observations $\hat{\mathbf{r}}$ are formulated as follows:

$$p(\hat{\mathbf{r}}|\mathbf{z}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_i} \times \exp\Bigg( -\frac{(\hat{r}_i - (\sqrt{x^2+y^2} + \sqrt{(x-a_i)^2+(y-b_i)^2}))^2}{2\sigma_i^2}\Bigg), \quad (3)$$

where $\mathbf{z} = [x, y]$.

## III. TSEM Method

In this section, we present the two steps of TSEM and summarize them in **Algorithm 1**. In the first step of TSEM, we assume $x$, $y$, $\sqrt{x^2+y^2}$ are independent of each other, and obtain temporary results for the target location based on this assumption. In the second step, we remove the assumption and update the estimation results. This idea is similar in spirit to the algorithm of [11] for TDOA positioning.

### *A. Step 1 of TSEM*

We can rewrite (1) as

$$2a_i x + 2b_i y - 2r_i\sqrt{x^2+y^2} = a_i^2 + b_i^2 - r_i^2.$$

Since $r_i = \hat{r}_i + e_i$, it follows that

$$-\frac{a_i^2+b_i^2-\hat{r}_i^2}{2} + a_i x + b_i y - \hat{r}_i\sqrt{x^2+y^2} = e_i(\sqrt{x^2+y^2} - \hat{r}_i) - \frac{e_i^2}{2}. \quad (4)$$

If $e_i$ is small, we can omit the second-order term $\frac{e_i^2}{2}$. Since there are $M$ such equations, we can express them in a matrix form as follows

$$\mathbf{h} - \mathbf{S}\boldsymbol{\theta} = \mathbf{B}\mathbf{e}, \tag{5}$$

where

$$\mathbf{h} = \begin{bmatrix} -\frac{a_1^2+b_1^2-\hat{r}_1^2}{2} \\ -\frac{a_2^2+b_2^2-\hat{r}_2^2}{2} \\ \vdots \\ -\frac{a_M^2+b_M^2-\hat{r}_M^2}{2} \end{bmatrix}, \mathbf{S} = -\begin{bmatrix} a_1 & b_1 & -\hat{r}_1 \\ a_2 & b_2 & -\hat{r}_2 \\ & \vdots & \\ a_M & b_M & -\hat{r}_M \end{bmatrix},$$

$\boldsymbol{\theta} = [x, y, \sqrt{x^2+y^2}]^T$, $\mathbf{e} = [e_1, e_2, \dots, e_M]^T$ and

$$\mathbf{B} = \text{diag}\ [\sqrt{x^2+y^2} - r_1, \sqrt{x^2+y^2} - r_2, \dots, \sqrt{x^2+y^2} - r_M] \tag{6}$$

where "diag" denotes the diagonal matrix. For notational convenience, we define the error vector

$$\boldsymbol{\varphi} = \mathbf{h} - \mathbf{S}\boldsymbol{\theta}. \tag{7}$$

The variance of $\boldsymbol{\varphi}$ is

$$\boldsymbol{\Psi} = E(\boldsymbol{\varphi}\boldsymbol{\varphi}^T) = \mathbf{B}\mathbf{Q}\mathbf{B}^T, \tag{8}$$

where $\mathbf{Q} = \text{diag}\ [\sigma_1^2, \sigma_2^2, \dots, \sigma_M^2]$. Because $\mathbf{B}$ depends on the true values $\mathbf{r}$ which is not obtainable, we use $\widehat{\mathbf{B}}$ (derived from the measurements $\hat{\mathbf{r}}$) in our calculations.

From equation (5) and the definition of $\boldsymbol{\varphi}$, it follows that $\boldsymbol{\varphi}$ is a vector of Gaussian variables; thus, the probability density function (pdf) of $\boldsymbol{\varphi}$ given $\boldsymbol{\theta}$ is

$$\begin{aligned} p(\boldsymbol{\varphi}|\boldsymbol{\theta}) &= \frac{1}{(2\pi)^{\frac{M}{2}}|\boldsymbol{\Psi}|^{\frac{1}{2}}} \exp(-\frac{1}{2}\boldsymbol{\varphi}^T\boldsymbol{\Psi}^{-1}\boldsymbol{\varphi}) \\ &= \frac{1}{(2\pi)^{\frac{M}{2}}|\boldsymbol{\Psi}|^{\frac{1}{2}}} \exp(-\frac{1}{2}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta})^T\boldsymbol{\Psi}^{-1}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta})). \end{aligned}$$

Then,

$$\begin{aligned} \ln\Big(p(\boldsymbol{\varphi}|\boldsymbol{\theta})\Big) &= -\frac{1}{2}\Big((\mathbf{h}-\mathbf{S}\boldsymbol{\theta})^T\boldsymbol{\Psi}^{-1}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta}) \\ &\quad + \ln|\boldsymbol{\Psi}|\Big) - \frac{M}{2}\ln 2\pi \end{aligned} \tag{9}$$

We *assume* for the moment that $x$, $y$, $\sqrt{x^2+y^2}$ are independent of each other (this - clearly non-fulfilled - assumption will be removed in the second step of the algorithm). Then, according to (9), the optimum $\boldsymbol{\theta}$ that maximizes $p(\boldsymbol{\varphi}|\boldsymbol{\theta})$ is equivalent to the one minimizing $\Pi = (\mathbf{h}-\mathbf{S}\boldsymbol{\theta})^T\boldsymbol{\Psi}^{-1}(\mathbf{h}-\mathbf{S}\boldsymbol{\theta}) + \ln|\boldsymbol{\Psi}|$.

If $\boldsymbol{\Psi}$ were a constant, the optimum $\boldsymbol{\theta}$ to minimize $\Pi$ satisfied $\frac{d\Pi}{d\boldsymbol{\theta}} = 0$. Taking th derivative of $\Pi$ over $\boldsymbol{\theta}$, we have

$$\frac{d\Pi}{d\boldsymbol{\theta}} = -2\mathbf{S}^T\boldsymbol{\Psi}^{-1}\mathbf{h} + 2\mathbf{S}^T\boldsymbol{\Psi}^{-1}\mathbf{S}\boldsymbol{\theta}.$$

Thus, the optimum $\boldsymbol{\theta}$ satisfies

$$\hat{\boldsymbol{\theta}} = arg\ min\{\Pi\} = (\mathbf{S}^T\boldsymbol{\Psi}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\boldsymbol{\Psi}^{-1}\mathbf{h}, \tag{10}$$

which provides $[\hat{x}, \hat{y}]$.

For our problem, $\boldsymbol{\Psi}$ is not a constant, but rather a function of $\boldsymbol{\theta}$ since $\mathbf{B}$ depends on the (unknown) values $[x, y]$. For this reason, the maximum-likelihood (ML) estimation method in (10) can not be directly used. To solve this problem, we employ the Expectation Maximization (EM) algorithm [16] to find the optimum $\boldsymbol{\theta}$. As shown in Fig. 2, the EM algorithms consists of two steps: an expectation step (E-step) and a maximization step (M-step). In the E-step, the missing data ($\boldsymbol{\Psi}$) is calculated given the estimate of parameters ($\boldsymbol{\theta}$). Note that $\boldsymbol{\theta}$ provides the values of $[x, y]$ and thus the value of $\widehat{\mathbf{B}}$, therefore, $\boldsymbol{\Psi}$ can be calculated using $\boldsymbol{\theta}$ by (8). In the M-step, the parameters ($\boldsymbol{\theta}$) are updated according to (10) to maximize the likelihood function (which is equivalent to minimizing $\Pi$). These two steps are iterated until convergence. Simulations in Section V show that commonly one iteration is enough for TSEM to closely approach the CRLB, which indicates that the global optimum is reached.

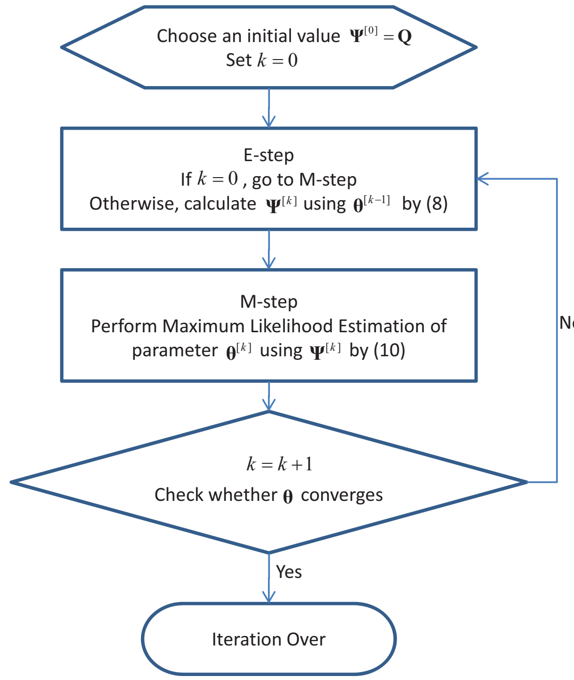



Figure 2. Illustration of EM algorithm for estimation of $\boldsymbol{\theta}$ in step 1 of TSEM

## B. Step 2 of TSEM

In the above calculations, $\hat{\boldsymbol{\theta}}$ contains three components $\hat{x}$, $\hat{y}$ and $\sqrt{\hat{x}^2+\hat{y}^2}$. They were previously assumed to be independent; however, $x$ and $y$ are clearly not independent of $\sqrt{\hat{x}^2+\hat{y}^2}$. As a matter of fact, we wish to eliminate the $\sqrt{\hat{x}^2+\hat{y}^2}$; this will be achieved in the following. Let

$$\hat{x} = x + n_1,\ \hat{y} = y + n_2 \text{ and } \sqrt{\hat{x}^2+\hat{y}^2} = \sqrt{x^2+y^2} + n_3, \tag{11}$$

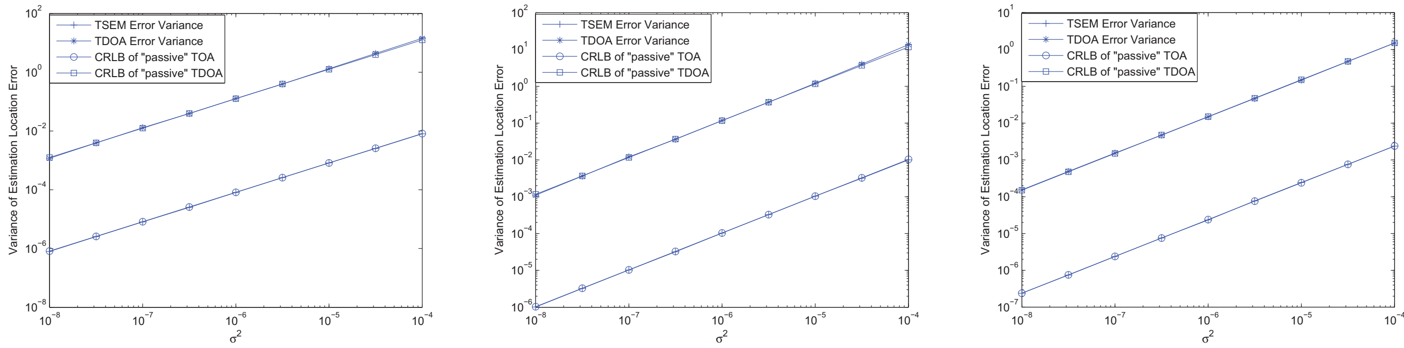
where $n_i (i = 1, 2, 3)$ is the estimation errors of the first step. Before proceeding forward, we need following Lemma.

*Lemma 1:* The covariance of $\hat{\boldsymbol{\theta}}$ can be expressed as

$$cov(\hat{\boldsymbol{\theta}}) = cov(\mathbf{n}^T\mathbf{n}) = (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}. \tag{12}$$

where $\mathbf{n} = [n_1, n_2, n_3]$., and $\boldsymbol{\Psi}$ and $\bar{\mathbf{S}}$ use the true/mean values of $x$, $y$, and $r_i$.

*Proof:* Please refer to the Appendix. ■




(a) (b) (c)

Figure 3. Illustration of coordinate transformation: (a) Simulation results of TSEM for the first set of configuration; (b) Simulation results of TSEM for the second set of configuration; (c) Simulation results of TSEM for the third set of configuration.

Note that since the true values of $x$, $y$, and $r_i$ are not obtainable, we use the estimated/measured values in the calculation of $cov(\hat{\boldsymbol{\theta}})$.

We can construct a vector $\mathbf{g}$ as follows

$$\mathbf{g} = \hat{\mathbf{\Theta}} - \mathbf{G}\Upsilon, \tag{13}$$

where $\hat{\mathbf{\Theta}} = [\hat{x}^2, \hat{y}^2, \hat{x}^2 + \hat{y}^2]^T$, $\Upsilon = [x^2, y^2]^T$ and

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Note that here $\hat{\mathbf{\Theta}}$ depends on the estimation results from the first step containing the estimated values $\hat{x}$ and $\hat{y}$ and $\Upsilon$ is the vector to be estimated. If $\hat{\mathbf{\Theta}}$ is obtained without error, $\mathbf{g} = 0$ and the location of the target is perfectly obtained. However, the error inevitably exists and we need to estimate $\Upsilon$. Then, substituting (11) into (13), omitting the second-order terms, it follows that,

$$\mathbf{g} = \begin{bmatrix} 2xn_1 \\ 2yn_2 \\ 2\sqrt{x^2 + y^2}n_3 \end{bmatrix}.$$

Besides,

$$\mathbf{\Omega} = E(\mathbf{g}\mathbf{g}^T) = 4\mathbf{D}cov(\hat{\boldsymbol{\theta}})\mathbf{D}, \tag{14}$$

where $\mathbf{D} = \text{diag}[x, y, \sqrt{x^2 + y^2}]$. Since $x$, $y$ are not known, $\mathbf{D}$ is calculated as $\hat{\mathbf{D}}$ using the estimated values $\hat{x}$, $\hat{y}$ from the first step. $\mathbf{g}$ can be approximated as a vector of Gaussian variables. Thus the maximum likelihood estimation of $\Upsilon$ is the one minimizing $(\hat{\mathbf{\Theta}} - \mathbf{G}\Upsilon)^T\mathbf{\Omega}^{-1}(\hat{\mathbf{\Theta}} - \mathbf{G}\Upsilon)$, expressed by

$$\hat{\Upsilon} = (\mathbf{G}^T\mathbf{\Omega}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{\Omega}^{-1}\hat{\mathbf{\Theta}}. \tag{15}$$

The value of $\mathbf{\Omega}$ is calculated according to (14) using the values of $\hat{x}$ and $\hat{y}$ in the first step. Finally, the estimation of target location $\mathbf{z}$ is obtained by

$$\hat{\mathbf{z}} = [\hat{x}, \hat{y}] = [\pm\sqrt{\hat{\Upsilon}_1}, \pm\sqrt{\hat{\Upsilon}_2}], \tag{16}$$

where $\hat{\Upsilon}_i$is the $i$th item of $\Upsilon$, $i = 1, 2$. To choose the correct one among the four values in (16), we can test the square error as follows

$$\chi = \sum_{i=1}^{M}(\sqrt{\hat{x}^2 + \hat{y}^2} + \sqrt{(\hat{x} - a_i)^2 + (\hat{y} - b_i)} - \hat{r}_i)^2. \tag{17}$$

The value of $\mathbf{z}$ that minimizes $\chi$ is considered as the final estimation result of the target location.

To summarize, the TSEM is given as follows:

**Algorithm 1** TSEM Location Estimation Method

1. Use EM algorithm as shown in Fig. 2 to obtain $\hat{\boldsymbol{\theta}}$,
2. Use the values of $\hat{x}$ and $\hat{y}$ from $\hat{\boldsymbol{\theta}}$, calculate $\mathbf{\Omega}$. Then, calculate the value of $\hat{\Upsilon}$ by (15),
3. Among the four potential values of $\hat{\mathbf{z}} = [\hat{x}, \hat{y}]$ obtained by (16), choose the one minimizing (17) as the final estimation for target location.

Note that the receiver locations should not all lie on a single line when using TSEM, because in this case $(\mathbf{S}^T\mathbf{\Psi}^{-1}\mathbf{S})^{-1}$ can become nearly singular, and the estimation process in (10) is not accurate.

We furthermore note that the algorithm achieves the CRLB for TOA algorithms. For space reasons, we omit the derivation of the CRLB of TOA (which, to the best of our knowledge, had not been available in the literature), and the proof that TSEM achieves the CRLB; details can be found in [17].

## IV. Simulation Results

To support our theoretical analysis, three sets of system configurations are simulated. In the first one, there are four receivers with locations at $[1, 0], [-1, 0], [1, 2], [-1, 2]$; the location of the transmitter is $[0, 0]$, and the target location is $[15, 11]$. The signal travel distance measurement variances at the receivers are identical, where the value of $\sigma^2$ is shown in the figures of simulation results. In the second configuration, the receiver and target locations are the same as the first set, but now the the signal travel distance measurement variances at the four receivers are $0.4\sigma^2$, $2\sigma^2$, $\sigma^2$, $0.4\sigma^2$, respectively. In the third set of simulation, there are six receivers with locations at $[10, 0], [-10, 0], [1, 2], [-1, 2], [30, 30], [-30, 30]$, the transmitter is at $[0, 0]$ and the target location is at $[94, 88]$. The error variances at the receivers are $1.9\sigma^2$, $3\sigma^2$, $0.4\sigma^2$, $0.9\sigma^2$, $1.2\sigma^2$ and $2.1\sigma^2$. Figs. 3 (a), Figs. 3 (b) and Fig. 3 (c) show the simulation results of first, second and third configurations, respectively.[2]

The figures furthermore compare the simulation results with the CRLBs of TDOA and TOA algorithsm in the figures. It can be observed that

1) The TSEM can closely approach the CRLB of TOA.
2) The CRLB of TDOA is much higher (can be about 30dB) than that of TOA.

## V. Conclusions

This paper proposes a novel algorithm, called TSEM, which exploits highly accurate range measurements of UWB signals. In contrast to previous papers, it considers the positioning scenario where the target is *not* the transmitter while at the same time the TOA (instead of just TDOA) information are available. The TSEM proceeds in two steps, first computing a set of estimates for a parameter set $x$, $y$, and $\sqrt{x^2+y^2}$ using the EM algorithm, second refining the location vector $[x, y]$ from the results of first step.

Numerical simulation results show that the estimated target location error variance of TSEM achieves the CRLB of TOA positioning algorithms. Further results show the CRLB of passive TOA estimation is much lower than that of TDOA. This indicates that with the TSME, the synchronization between the transmitter and the receivers can substantially decrease the localization error.

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## Appendix
## Proof of Lemma 1

Let $\hat{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}} + \mathbf{n}$, $\hat{\mathbf{S}} = \bar{\mathbf{S}} + \mathbf{e_S}$ and $\hat{\mathbf{h}} = \bar{\mathbf{h}} + \mathbf{e_h}$, where $\bar{\boldsymbol{\theta}}$, $\bar{\mathbf{S}}$ and $\bar{\mathbf{h}}$ are true/mean values and $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{S}}$ and $\hat{\mathbf{h}}$ are the measured/estimated values. Obviously,

$$\bar{\mathbf{h}} - \bar{\mathbf{S}}\bar{\boldsymbol{\theta}} = 0. \tag{18}$$

According to (5) and (7),

$$\begin{aligned}\boldsymbol{\varphi} &= \bar{\mathbf{h}} + \mathbf{e_h} - (\bar{\mathbf{S}} + \mathbf{e_S})\bar{\boldsymbol{\theta}} \\ &= \mathbf{e_h} - \mathbf{e_S}\bar{\boldsymbol{\theta}}. \end{aligned} \tag{19}$$

Multiply both sides of (10) by $(\bar{\mathbf{S}}^T + \mathbf{e_S}^T)\boldsymbol{\Psi}^{-1}(\bar{\mathbf{S}} + \mathbf{e_S})$, it follows

$$\begin{aligned}&(\bar{\mathbf{S}}^T + \mathbf{e_S}^T)\boldsymbol{\Psi}^{-1}(\bar{\mathbf{S}} + \mathbf{e_S})(\bar{\boldsymbol{\theta}} + \mathbf{n}) \\ &= (\bar{\mathbf{S}}^T + \mathbf{e_S}^T)\boldsymbol{\Psi}^{-1}(\bar{\mathbf{h}} + \mathbf{e_h}),\end{aligned}$$

leave only the liner perturbation items by omitting the second order errors, using (18), it follows that

$$\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}}\mathbf{n} = \bar{\mathbf{S}}\boldsymbol{\Psi}^{-1}(\mathbf{e_h} - \mathbf{e_S}\bar{\boldsymbol{\theta}}).$$

Then, we obtain

$$\begin{aligned}\mathbf{n} &= (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}\boldsymbol{\Psi}^{-1}(\mathbf{e_h} - \mathbf{e_S}\bar{\boldsymbol{\theta}}) \\ &= (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}\boldsymbol{\Psi}^{-1}\boldsymbol{\varphi}.\end{aligned} \tag{20}$$

[2]During the simulations, only one iteration is used for the calculation of $\mathbf{B}$.

According to (5) and (7), substitute $\boldsymbol{\varphi}$ by $\mathbf{Be}$, it follows that

$$\mathbf{n} = (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}\boldsymbol{\Psi}^{-1}\mathbf{Be}.$$

Then,

$$\begin{aligned}cov(\hat{\boldsymbol{\theta}}) &= E(\mathbf{nn}^T) \\ &= (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\mathbf{B}E(\mathbf{ee}^T) \\ &\quad \times\mathbf{B^T\Psi^{-1}}\bar{\mathbf{S}}[(\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}]^T.\end{aligned}$$

Because $\mathbf{B}E(\mathbf{ee}^T)\mathbf{B^T} = \boldsymbol{\Psi}$, and $[(\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}]^T = (\bar{\mathbf{S}}^T\bar{\boldsymbol{\Psi}}^{-1}\bar{\mathbf{S}})^{-1}$, it follows that

$$\begin{aligned}cov(\hat{\boldsymbol{\theta}}) &= (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1}\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}}(\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1} \\ &= (\bar{\mathbf{S}}^T\boldsymbol{\Psi}^{-1}\bar{\mathbf{S}})^{-1},\end{aligned}$$

which ends the proof of Lemma 1.

## References

[1] E. Paolini, A. Giorgetti, M. Chiani, R. Minutolo, and M. Montanari, “Localization capability of cooperative anti-intruder radar systems,” *EURASIP Journal on Advances in Signal Processing*, vol. 2008, no. 17, 2008.

[2] S. Gezici, Z. Tian, G. Giannakis, H. Kobayashi, A. Molisch, H. Poor, and Z. Sahinoglu, “Localization via ultra-wideband radios: a look at positioning aspects for future sensor networks,” *IEEE Signal Processing Magazine*, vol. 22, no. 4, pp. 70 – 84, July 2005.

[3] J. Zhang, P. Orlik, Z. Sahinoglu, A. Molisch, and P. Kinney, “UWB Systems for Wireless Sensor Networks,” *Proceedings of the IEEE*, vol. 97, no. 2, pp. 313 –331, 2009.

[4] M. Win, A. Conti, S. Mazuelas, Y. Shen, W. Gifford, D. Dardari, and M. Chiani, “Network localization and navigation via cooperation,” *Communications Magazine, IEEE*, vol. 49, no. 5, pp. 56 –62, may 2011.

[5] A. Weiss, “On the accuracy of a cellular location system based on RSS measurements,” *Vehicular Technology, IEEE Transactions on*, vol. 52, no. 6, pp. 1508 – 1518, nov. 2003.

[6] I. Guvenc and C.-C. Chong, “A Survey on TOA Based Wireless Localization and NLOS Mitigation Techniques,” *IEEE Communications Surveys & Tutorials*, vol. 11, no. 3, pp. 107–124, Quarter 2009.

[7] J. Xu, M. Ma, and C. Law, “Position Estimation Using UWB TDOA Measurements,” in *The 2006 IEEE 2006 International Conference on Ultra-Wideband*, sept. 2006, pp. 605 –610.

[8] K. Yu, J. Montillet, A. Rabbachin, P. Cheong, and I. Oppermann, “UWB location and tracking for wireless embedded networks,” *Signal Processing*, vol. 86, no. 9, pp. 2153–2171, 2006.

[9] R. Zetik, J. Sachs, and R. Thoma, “UWB short-range radar sensing - The architecture of a baseband, pseudo-noise UWB radar sensor,” *IEEE Instrumentation Measurement Magazine*, vol. 10, no. 2, pp. 39 –45, april 2007.

[10] C.-C. Chui and R. Scholtz, “Time transfer in impulse radio networks,” *IEEE Trans. on Communications*, vol. 57, no. 9, pp. 2771 –2781, 2009.

[11] Y. Chan and K. Ho, “A simple and efficient estimator for hyperbolic location,” *IEEE Transactions on Signal Processing*, vol. 42, no. 8, pp. 1905–1915, 1994.

[12] Y.-T. Chan, H. Yau Chin Hang, and P. chung Ching, “Exact and approximate maximum likelihood localization algorithms,” *IEEE Transactions on Vehicular Technology*, vol. 55, no. 1, pp. 10–16, Jan. 2006.

[13] K. Cheung, H. So, W.-K. Ma, and Y. Chan, “Least squares algorithms for time-of-arrival-based mobile location,” *IEEE Transactions on Signal Processing*, vol. 52, no. 4, pp. 1121 – 1130, april 2004.

[14] J. Salmi, S. Sangodoyin, and A. F. Molisch, “High resolution parameter estimation for ultra-wideband MIMO radar,” in *Proceedings of The 44nd Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, CA, Nov. 7-10 2010.

[15] D. Dardari, A. Conti, U. Ferner, A. Giorgetti, and M. Win, “Ranging With Ultrawide Bandwidth Signals in Multipath Environments,” *Proceedings of the IEEE*, vol. 97, no. 2, pp. 404 –426, Feb. 2009.

[16] T. Moon, “The expectation-maximization algorithm,” *IEEE Signal Processing Magazine*, vol. 13, no. 6, pp. 47 –60, nov 1996.

[17] J. Shen, A. F. Molisch, and J. Salmi, “Accurate Passive Location Estimation Using TOA Measurements,” *submitted to IEEE Transactions on Wireless Communications*, 2011.