On the Physical Limitations of the Interaction of a Spherical Aperture and a Random Field

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***Abstract*—This paper derives physical limitations on the interactions of antennas exciting TM or TE modes (but not both) and wireless propagation channels. The derivation is based on the spherical vector wave expansion of the electromagnetic field outside a sphere circumscribing the antennas. The result is an extension of the seminal work of Chu on the classical limitations on maximum antenna gain and radiation Q. Rather than maximizing antenna gain in a single direction we obtain physical limitations on the antenna gain pattern, which is directly translated to more condensed parameters, i.e., the instantaneous effective gain G_{i} and the mean effective gain G_{e} if instantaneous realizations or correlation statistics of the expansion coefficients of the electromagnetic field are known, respectively. The obtained limitations are on the maximum of G_{i}/Q and G_{e}/Q, which establish a trade-off between link gain and Q.**

***Index Terms*—Mean effective gain, physical bounds, quality factor, spherical vector waves.**

I. Introduction

BANDWIDTH is a valuable resource. In wireless communication systems it can be employed to provide high data rates and/or to accommodate several communication standards operating over a wide range of frequencies on the same, commonly small, communication device such as a wireless handheld terminal. Antennas are therefore required to exhibit large bandwidths while occupying a small volume. This is a challenging requirement ruled by physical limitations. It is well-known that the radiation properties of an antenna are related to its size [1], [2]. For example, the radiation Q, which is defined as the ratio of the power stored by the reactive field of an antenna to the power loss, considerably increases as the electrical size[1] of the antenna decreases [2]. For narrowband antennas, radiation Q is inversely proportional to the fractional bandwidth of the antenna.[2] Thus, a high antenna Q is highly undesirable, since it leads to a narrow impedance bandwidth for electrically small antennas as well as poor radiation efficiency due to high ohmic and dielectric losses.

Physical performance limits of antennas were initially established by Wheeler, [1], and Chu, [2]. In his work, Chu derived the lowest possible radiation Q, the maximum gain and the maximum possible gain-to-Q ratio for linearly polarized omnidirectional antennas using the spherical vector wave expansion of the electromagnetic field outside the sphere of minimum radius that completely encloses the antenna. Since then, this problem has drawn the attention of many researchers, [3]–[12] with a summary in [13]. Recently, Chu's classical results have been refined by new, more precise and general performance limits that depend upon the shapes and materials of the antennas [14].

Traditionally, investigation of antenna performance limits have involved either (i) the maximization of the antenna gain, G, in some specific direction or (ii) the minimization of the radiation Q, or (iii) the maximization of the ratio between them, G/Q, [2], [9]. The latter criterion provides the condition for the minimum Q to achieve a certain gain or as the condition for the maximum gain achievable at a given Q. Hence, the maximum ratio G/Q provides a compromise between gain and bandwidth since Q is roughly proportional to the inverse of the antenna bandwidth [15].

In most cellular and wireless LAN systems, the mean effective gain (MEG) is a more important quantity than the maximum antenna gain. This can be understood as follows: In communication links with a pronounced line-of-sight (LOS) propagation path between the receiver and the transmitter, the antenna gain is indeed a good measure of the communication efficiency of the antenna, which can be assessed from the Friis equation [16], [17], and it is clear that the gain should be maximized into the direction of the LOS component. On the other hand, in multipath propagation channels with no dominant component (non-line-of-sight, NLOS), the maximization of the antenna gain in a single or specific direction is of less relevance. Rather, we prefer antennas that are capable of receiving all relevant multipath components. Hence, for maximizing performance in NLOS scenarios, antenna gain is not equally efficient as a figure of merit of an antenna. We need instead a description that takes into account the strength of the multipath components (MPCs)

Manuscript received February 19, 2009; revised November 27, 2009; accepted March 18, 2010. Date of publication November 09, 2010; date of current version January 04, 2011. This work was supported in part by the SSF High Speed Wireless Center and in part by an INGVAR grant from the Swedish Foundation for Strategic Research.

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Digital Object Identifier 10.1109/TAP.2010.2090639

[1]The electrical size is defined as the product ka, where k is the wave-number and a is the radius of the smallest sphere circumscribing the antenna.

[2]For antennas with very large bandwidth, and thus $Q < 1$, a direct relationship is difficult to define. Therefore, following Chu's approach in [2], we always use $\max\{Q, 1\}$ rather than Q to evaluate the radiation quality of an antenna.

in the different directions, as well as the antenna gain in all those directions. In this case the radiation gain pattern of the antenna and the radiation efficiency of the antenna[3] together with the polarization-dependent, angular distribution of the MPC strength provide a more useful description. However, such a full characterization of antennas becomes too cumbersome for most practical purposes. For this reason, the Mean Effective Gain (MEG) (see, e.g., [18]–[20] for further discussion on MEG) is a more useful measure of the link quality in a given propagation environment. By definition the MEG incorporates parameters that describe both the antenna and the propagation channel. Essentially, its definition is based on the partial antenna gain patterns[4] weighted by the power-angular spectrum (PAS) of the two orthogonal polarizations, respectively, and the cross-polarization ratio (XPR) of the propagation channel.[5] We can actually distinguish between two link gain parameters, namely the MEG, $G_{\rm e}$, which weights the gain pattern by the average PAS and XPR, and the "instantaneous" effective gain, $G_{\rm i}$, which weights the pattern by a realization of the (stochastically varying) channel. Since most propagation channels in today's wireless systems are NLOS, rather than maximizing the antenna gain in a specific direction, we aim at obtaining the maximum MEG and the maximum instantaneous gain.

In [20] we showed that both $G_{\rm i}$ and $G_{\rm e}$ are maximized when the receive (or transmit) antenna coefficients equal the complex conjugates of the expansion coefficients of the incoming field in spherical vector waves. In this case, the maximum $G_{\rm i}$ and $G_{\rm e}$ are bounded by $4\pi\sum\eta_i$, where η_i stands for the radiation efficiency of the antenna port i. Conjugate mode-matching provides a maximization of the link gain performance of an antenna in a multipath channel, however, without taking into account the physical limitations imposed by the antennas. Hence, the maximization of $G_{\rm i}$ and $G_{\rm e}$ was made independently of bandwidth constraints, which will result in rather narrowband antennas.

In this paper we therefore generalize results obtained by Chu and Wheeler to antennas in multi-path propagation channels. We do this by using the spherical vector wave expansion of the electromagnetic field outside the minimum sphere enclosing the antenna, [20]. The obtained limitations are on the maximum of $G_{\rm i}/Q$ and $G_{\rm e}/Q$, which establish a trade-off between link gain and Q, i.e., they provide the maximum $G_{\rm i}$ or maximum $G_{\rm e}$ for a given Q or the minimum Q for given $G_{\rm i}$ or $G_{\rm e}$. The main findings are summarized as follows:

1) If realizations of the channel are known, the transmission coefficients (or reception coefficients for reciprocal antennas) that optimize $G_{\rm i}/Q$ are given by the complex conjugate of the spherical vector wave expansion coefficients of the field impinging at the antenna, $\boldsymbol{a}_\tau$, weighted by the inverse of the radiation quality Q of the mode of order l, Q_l. Thus, the contribution of higher order modes will be attenuated (filtered out), because Q_l increases with the mode order l. The corresponding $G_{\rm i}$ and Q both depend on the realizations of $\boldsymbol{a}_\tau$ meaning that both the antenna gain pattern and the bandwidth of the antenna must change adaptively. On the other hand for electrically small antennas the optimal bandwidth coincides with Chu's predictions and is independent of the channel, while the antenna gain pattern (dipole modes) must still be adaptively changed. Electrically small antennas are the most efficient ones in terms of the use of the available channel modes.
2) If only the correlation matrix of the channel is known, the transmission coefficients (or reception coefficients for reciprocal antennas) that optimize $G_{\rm e}/Q$ are given by the eigenvector corresponding to the largest eigenvalue of the correlation matrix of the spherical vector wave expansion coefficients of the field impinging at the antenna, $\boldsymbol{\mathcal{R}}_{a_\tau}$, weighted by the inverse of the radiation Q of the mode of order l, Q_l. Here again, the contribution of higher-order modes will be attenuated (filtered out) as Q_l increases with the mode order l. The corresponding $G_{\rm e}$ and Q both depend on the correlation matrix of $\boldsymbol{a}_\tau$. For electrically small antennas the optimal bandwidth coincides with Chu's predictions and is independent of the correlation properties of the channel. On the other hand, the optimal antenna gain pattern (dipole modes) depends on the correlation properties of the channel. Electrically small antennas are the most efficient with respect to the use of the available channel modes in this case too.
3) The optimal performance of multi-port antenna systems with no mutual coupling, i.e., non-interacting ports is dictated by the optimal performance of the single-port antenna case since each port must have identical performance.

The remainder of the paper is organized as follows. Section II presents a brief introduction to the spherical vector wave expansion of the electromagnetic field, the antenna scattering matrix, the mean effective gain, the instantaneous effective gain and the radiation Q of the antenna. In Section III we state and solve the maximization problem of $G_{\rm e}/Q$ and $G_{\rm i}/Q$ for antennas enclosed in a spherical volume. Here, we also present numerical results that illustrate our results based on a generic propagation channel. The conclusions are provided in Section IV.

II. Spherical Vector Wave Expansion of an Antenna, the Propagation Channel Fields and Related Parameters

In [20] we developed a formalism for analyzing the interaction between the antennas and the propagation channel, where the spherical vector wave expansion of the electromagnetic field and the scattering matrix were employed as the two main modeling tools. We present next the main points of these tools.

Consider an antenna system enclosed by an (imaginary) sphere of radius a. The electric field, $\boldsymbol{E}(\boldsymbol{r})$, outside this sphere can be expanded in *outgoing spherical vector waves* $\boldsymbol{u}_\iota(k\boldsymbol{r})$ and *incoming spherical vector waves* $\boldsymbol{v}_\iota(k\boldsymbol{r})$ as, [21], [22]

$$\boldsymbol{E}(\boldsymbol{r}) = k\sqrt{2\eta}\sum_\iota a_\iota\boldsymbol{v}_\iota(k\boldsymbol{r}) + b_\iota\boldsymbol{u}_\iota(k\boldsymbol{r}),\ \text{for } |\boldsymbol{r}|\geq a \qquad (1)$$

[3]i.e., how much of the input power at some reference plane or point is actually radiated by the antenna.

[4]The partial antenna gain patterns are defined in two orthogonal polarizations, their sum being equal to the total antenna gain pattern. For 100% effective antennas the antenna gain pattern equals the antenna directivity.

[5]i.e., the ratio of the power in the θ-polarization to the power in the ϕ-polarization.

where $\iota = (\tau m l)$ is the multi-index identified with the number $\iota = 2(l^2 + l - 1 + m) + \tau$, k is the wave-number and η is the free-space impedance; see Appendix A for a brief discussion of spherical vector waves.

The scattering matrix of an N-port antenna provides a full description of all its properties [22]. The scattering matrix relates the incoming signals, $\boldsymbol{v} \in \mathbb{C}^{N\times 1}$ and waves, $\boldsymbol{a} \in \mathbb{C}^{\infty\times 1}$, the outgoing signals, $\boldsymbol{w} \in \mathbb{C}^{N\times 1}$ and waves $\boldsymbol{b} \in \mathbb{C}^{\infty\times 1}$, the matrix containing the complex antenna reflection coefficients, $\boldsymbol{\Gamma} \in \mathbb{C}^{N\times N}$, the matrix containing the antenna receiving coefficients, $\mathbf{R} \in \mathbb{C}^{N\times\infty}$, the matrix containing the antenna transmitting coefficients, $\mathbf{T} \in \mathbb{C}^{\infty\times N}$ and the matrix containing the antenna scattering coefficients, $\mathbf{S} \in \mathbb{C}^{\infty\times\infty}$

$$\begin{pmatrix} \boldsymbol{\Gamma} & \mathbf{R} \\ \mathbf{T} & \mathbf{S} \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{a} \end{pmatrix} = \begin{pmatrix} \boldsymbol{w} \\ \boldsymbol{b} \end{pmatrix}. \tag{2}$$

For reciprocal antennas we also notice that the following relationship is valid [22]

$$R_{n,\tau m l} = (-1)^m T_{\tau(-m)l,n} \tag{3}$$

where $R_{n,\tau m l}$ and $T_{\tau m l,n}$ are elements of matrices $\mathbf{R}$ and $\mathbf{T}$, respectively. The received (outgoing) signals are given by

$$\boldsymbol{w} = \mathbf{R}\boldsymbol{a} \tag{4}$$

while the transmitted signals are related to the outgoing waves as

$$\boldsymbol{b} = \mathbf{T}\boldsymbol{v}. \tag{5}$$

A. "Instantaneous" Effective Gain and Mean Effective Gain

Consider now a multi-port antenna system with N ports in receive mode. Assume further that the propagation channel[6] is characterized by a random process as described in [20]. Then, we can compute the "instantaneous" effective gain as

$$G_{\mathrm{i}} = \frac{\mathrm{tr}\{\mathbf{R}\boldsymbol{a}\boldsymbol{a}^\dagger\mathbf{R}^\dagger\}}{\mathrm{tr}\{\boldsymbol{a}\boldsymbol{a}^\dagger\}} \tag{6}$$

where $\mathrm{tr}\{\mathbf{R}\boldsymbol{a}\boldsymbol{a}^\dagger\mathbf{R}^\dagger\} = \|\mathbf{R}\boldsymbol{a}\|_F^2 = \|\boldsymbol{w}\|_F^2$ is the "instantaneous" received (or link) power and $\mathrm{tr}\{\boldsymbol{a}\boldsymbol{a}^\dagger\} = \|\boldsymbol{a}\|_F^2$ is the "instantaneous" power of the available electromagnetic signals, where $\|.\|_F^2$ is the Frobenius norm. The symbol $(.)^\dagger$ denotes Hermitian transpose.

Besides the instantaneous power it is also relevant to quantify the average received power. The mean effective gain (MEG) [18], [19] defined in terms of the spherical vector wave expansion coefficients can be computed as [20]

$$G_{\mathrm{e}} = \frac{\mathrm{tr}\{\mathbf{R}\boldsymbol{\mathcal{R}}_a\mathbf{R}^\dagger\}}{\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}} \tag{7}$$

where $\boldsymbol{\mathcal{R}}_a$ is the mode correlation matrix and $\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\} = \langle\|\boldsymbol{a}\|_F^2\rangle$ is the average power[7] of the available electromagnetic signals and $\mathrm{tr}\{\mathbf{R}\boldsymbol{\mathcal{R}}_a\mathbf{R}^\dagger\} = \langle\|\boldsymbol{w}_F^2\rangle$ is the average received power; $\langle\rangle$ denotes expectation over the ensemble, and the different ensemble realizations can be interpreted as being taken over time, space or frequency. MEG is a measure of the communication efficiency of a given antenna over the multipath propagation channel [18]. It should be noticed that in general $G_{\mathrm{e}} \neq \langle G_{\mathrm{i}}\rangle$. The instantaneous effective gain is a measure of link efficiency that monitors the fast fading variation over a small-scale area while MEG is a measure of link efficiency averaged over the small-scale variation.

B. Radiation Q of Antennas

The field modes contribute to the radiated power and to the reactive power. Hence, each mode l is characterized by its own radiation quality factor[8] Q_l, which for a lossless antenna is defined as the ratio of the energy stored to the radiated energy [5]. The Q_l increases rapidly as the electrical size of the antenna becomes smaller than the mode number l, i.e., when $ka \lesssim l$, where a is the ratio of the minimum sphere enclosing the antenna and $1/k = \lambda/2\pi$ is the radius of the radiansphere [1]. The inverse of the radiation Q factor of an antenna is usually used as an estimate of the fractional bandwidth of an antenna, i.e., $\Delta f/f_0 \approx Q^{-1}$ for $Q \gg 1$, where f_0 is the center frequency and $\Delta f = f_{\mathrm{H}} - f_{\mathrm{L}}$ is the bandwidth expressed as the difference between the highest and the lowest frequency occupied by the antenna.

For a nonresonant antenna, which is tuned to resonance by a reactive element, i.e., the input impedance of the antenna becomes purely real, the radiation Q is defined as, [15]

$$Q = \frac{2\omega \max(W_{\mathrm{m}}, W_{\mathrm{e}})}{P} \tag{8}$$

where $\omega = 2\pi f$ is the angular frequency, W_{m} is the stored magnetic energy, W_{e} is the stored electric energy and P is the dissipated power, which for lossless antennas equals the radiated power P_{rad}. At the resonance frequency, the stored magnetic and electric field energies are equal, $W_{\mathrm{m}} = W_{\mathrm{e}}$. Analytic expressions for the radiation Q_l of mode l are given in [5], which are the same for both TM_{lm} $(\tau = 2)$ and TE_{lm} $(\tau = 1)$ modes independently of index m

$$\begin{aligned} Q_l = ka &- \left(\frac{(ka)^3}{2} + (l+1)ka\right)\left|\mathrm{h}_l^{(2)}(ka)\right|^2 \\ &- \frac{(ka)^3}{2}\left|\mathrm{h}_{l+1}^{(2)}(ka)\right|^2 \\ &+ (ka)^2\frac{2l+3}{2}\Re\left\{\mathrm{h}_l^{(2)}(ka)\mathrm{h}_{l+1}^{(2)*}(ka)\right\} \end{aligned} \tag{9}$$

where $\mathrm{h}_l^{(2)}$ denote the spherical Hankel function of the 2nd kind [25] and $\Re\{\mathrm{x}\}$ denotes the real part of x.

The definition of the radiation Q-factor in the case of a multiple-antenna system with multiple input ports or a number of separate antennas in close proximity is not as straightforward as for the single-port antenna case [2], [15]. As previously, we assume that the antenna system is circumscribed by a sphere. We further assume that there is no mutual coupling between

[6]It is worthwhile to notice that through the paper we assume that the spatial properties of the propagation channel are preserved over the considered bandwidth. This assumption is the more accurate the narrower the bandwidth considered and is sufficient for current communications systems [23], [24].

[7]A useful normalization of the average link power is obtained from the energy conservation law and the addition law of spherical vector harmonics.

[8]The Q of any mode given by the multi-index $(\tau m l)$ depends only on l.

the antenna ports. This situation is often desirable when designing wireless communication systems since it usually implies low signal correlation and therefore increased diversity performance. Hence, the multi-port antenna system is basically a single radiating system.[9] Thus, the Q-factor is computed as the ratio of the power stored by the reactive field of the whole antenna system to the sum of power losses from all antenna ports. Hence, the radiation Q factor of a vertically or horizontally polarized antenna can be expressed in terms of the expansion in spherical waves as

$$Q = \frac{\mathrm{tr}\mathbf{T}^\dagger \mathbf{Q}\mathbf{T}}{\mathrm{tr}\mathbf{T}^\dagger \mathbf{T}} \tag{10}$$

where the matrix $\mathbf{Q}$ is a diagonal matrix given by

$$\mathbf{Q} = \mathrm{diag}(Q_l) \tag{11}$$

where Q_l is given by (10) and $\mathbf{T}$ is the matrix containing the antenna transmitting coefficients. This definition represents an average behavior. For the single-port antenna case, i.e., $N=1$, when TM_{lm} $(\tau=2)$ or TE_{lm} $(\tau=1)$ modes are excited by the antenna, the radiation Q is then given by [5]

$$Q = \frac{\sum_{l=1}^{L}\sum_{m=-l}^{l} |t_{\tau ml}|^2 Q_l}{\sum_{l=1}^{L}\sum_{m=-l}^{l} |t_{\tau ml}|^2} \tag{12}$$

where we have introduced the notation $t_{\tau ml} = T_{\tau ml,1}$ for the transmission coefficient of the single-port antenna.

The Q-factor of the six lowest order modes is $Q=(ka)^{-3}+(ka)^{-1}$, which is the minimum achievable Q, when only one polarization is excited. On the other hand the combination of one TM_{1m} and one TE_{1m} mode gives a lower Q-factor, $Q=(ka)^{-3}/2+(ka)^{-1}$ [7].

We can now obtain the Q of a lossless antenna expanded in spherical vector waves from (10)–(12). The radiation Q is a property of the antenna and the fields related to it. The Q-factor does not always provide a perfect description of an antenna in terms of bandwidth, it however dictates antenna performance with a clear impact of antenna size on antenna gain. Moreover, the physical implications are straightforward, antennas with high Q-factors have a large amounts of reactive energy stored in the near zone. This in turn implies that coupling to electromagnetic objects in the near-field zone will produce high losses and in general large currents, narrow bandwidth, and large frequency sensitivity.

III. Optimum Antenna-Channel Interaction

We now proceed to derive the maximum of G_i/Q and G_e/Q of both multi-port antennas and single-port antennas when solely TM_{lm} $(\tau=2)$ or TE_{lm} $(\tau=1)$ modes are excited. It is straightforward to see that a general mathematical formulation of the problem reads as

$$\max \frac{\mathrm{tr}\{\mathbf{R}\boldsymbol{\mathcal{R}}\mathbf{R}^\dagger\}}{\mathrm{tr}\mathbf{R}\mathbf{Q}\mathbf{R}^\dagger} \tag{13}$$

where $\mathrm{tr}\{\mathbf{R}\mathbf{R}^\dagger\}=4\pi$, $\mathbf{Q}$ is given by (10) and $\boldsymbol{\mathcal{R}} = \boldsymbol{a}_\tau \boldsymbol{a}_\tau^\dagger$ when evaluating G_i/Q or $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}_{\boldsymbol{a}_\tau}$, when evaluating G_e/Q, where $\boldsymbol{a}_\tau$ are the field expansion coefficients corresponding to either TM or TE modes.

Observing that we have assumed multi-port antennas with no mutual-coupling between ports each antenna can now be "adapted" to the field independently from each other. Hence, the performance of each port in terms of G_i/Q or G_e/Q is identical to the single-port case, which is mathematically derived in Appendices B and C. Clearly, since the following relations are valid

$$\max G_\mathrm{i}/Q|_{\mathrm{multi-port}} = N \max G_\mathrm{i}/Q|_{\mathrm{single-port}} \tag{14}$$
$$\max G_\mathrm{e}/Q|_{\mathrm{multi-port}} = N \max G_\mathrm{e}/Q|_{\mathrm{single-port}} \tag{15}$$

where N is the number of ports it suffices to consider the single-port antenna case.

In the following we have chosen to restrict both the link power and the available power to the same range of modes $1 \le l \le L$. Our goal is therefore to study the interaction of an antenna that can sense field modes (TE or TM but not both) with maximum index l of at most L. Hence, the performance of an antenna is compared to the field exciting the same modes. In this case the performance, e.g., the link power obtained with an electrically small antenna can in fact be larger compared to the performance of larger antenna as we are going to see in the next sections. However, if the performance of two antennas are compared relative the same available power, then, obviously the larger antenna will perform better than or equal to the electrically small antenna under same conditions.

A. Maximum G_i/Q of Ideal Antennas

The transmission coefficients, $\boldsymbol{t} = (\mathbf{T})_{:,1}$ that maximize G_i/Q are obtained as (see Appendix B for a derivation)

$$\boldsymbol{t} = \left(\frac{4\pi}{\boldsymbol{a}_\tau^\dagger \mathbf{Q}^{-2}\boldsymbol{a}_\tau}\right)^{\frac{1}{2}} \mathbf{Q}^{-1}\boldsymbol{a}_\tau^* \tag{16}$$

where $\boldsymbol{a}_\tau^*$ is a vector containing the complex conjugated values of the expansion coefficients of the field impinging on the antenna, the index m has been interchanged with $-m$ in order to represent the link gain as a function of transmission coefficients instead of reception coefficients, the index τ takes on 1 or 2 depending on the type of antenna.

The optimum ratio G_i/Q with corresponding G_i and Q are given by

$$\max \frac{G_\mathrm{i}}{Q} = 4\pi \frac{\boldsymbol{a}_\tau^\dagger \mathbf{Q}^{-1}\boldsymbol{a}_\tau}{\boldsymbol{a}_\tau^\dagger \boldsymbol{a}_\tau} \tag{17}$$

$$G_\mathrm{i} = \frac{4\pi}{\boldsymbol{a}^\dagger \boldsymbol{a}} \frac{\left(\boldsymbol{a}_\tau^\dagger \mathbf{Q}^{-1}\boldsymbol{a}_\tau\right)^2}{\boldsymbol{a}_\tau^\dagger \mathbf{Q}^{-2}\boldsymbol{a}_\tau} \tag{18}$$

$$Q = \frac{\boldsymbol{a}_\tau^\dagger \mathbf{Q}^{-1}\boldsymbol{a}_\tau}{\boldsymbol{a}_\tau^\dagger \mathbf{Q}^{-2}\boldsymbol{a}_\tau}. \tag{19}$$

[9]An optional definition is more sensitive to the actual performance of specific antenna port. According to this definition the radiation Q is the average over each antenna port

$$Q = N^{-1}\sum_{i=1}^{N} \frac{(\mathbf{T})^\dagger_{:i}\mathbf{Q}(\mathbf{T})_{:i}}{(\mathbf{T})^\dagger_{:i}(\mathbf{T})_{:i}}$$

where $(\mathbf{T})_{:i}$ denotes the i^{th} column of the $\mathbf{T}$ matrix. Clearly, this definition perfectly coincides with (10) if the transmission vectors $(\mathbf{T})_{:i}$ are the same for all i.

We see from (16)–(19) that the radiation Q_l of the partial modes of the antenna have a filtering effect on the modes of the propagation channel, $\boldsymbol{a}_\tau^*$, i.e., the optimum transmission coefficients of an antenna, in the sense of the maximum ratio G_i/Q, in a propagation channel characterized by the expansion coefficients of the field impinging on the antenna, $\boldsymbol{a}_\tau$, are attenuated by the inverse of the radiation Q of the corresponding partial waves as the electric length ka increases for a multipole index l and vice versa.

Both G_i/Q and the corresponding G_i and Q are, in general, stochastic variables due to the stochastic nature of the propagation channel. Hence, the optimal antenna must adapt its transmission coefficients to the realizations of the propagation channel.[10] Obviously, optimal performance must be related to a specific propagation channel. In other words, when defining optimal performance of an antenna in a wireless propagation channel or optimal antenna-channel interaction, the statistics of the channel, $\boldsymbol{a}$, must be specified. It is also straightforward to see that for all $\mathbf{Q}$, $\boldsymbol{a}_\tau$, $\boldsymbol{a}$ and L it holds that $G_\mathrm{i} \leq 4\pi$ in agreement with [20].

For electrically small antennas ($ka \leq 0.5$)[11] the radiation field of the antenna will be dominated by the dipole modes ($l = 1$) and therefore only the first term Q_1 is of relevance, [2], [7], leading to the following result

$$Q \approx Q_1, \tag{20}$$

$$G_\mathrm{i} = 4\pi \frac{\sum_{m=-1}^{1} |a_{\tau m 1}|^2}{\sum_{\tau=1}^{2}\sum_{l=1}^{L}\sum_{m=-l}^{l} |a_{\tau m l}|^2} \tag{21}$$

$$\max \frac{G_\mathrm{i}}{Q} \approx \frac{4\pi}{Q_1} \frac{\sum_{m=-1}^{1} |a_{\tau m 1}|^2}{\sum_{\tau=1}^{2}\sum_{l=1}^{L}\sum_{m=-l}^{l} |a_{\tau m l}|^2}. \tag{22}$$

Several conclusions can be drawn from (20)–(22). First we see that for lossless, electrically small antennas exciting TM or TE modes only the radiation Q is identical to the radiation quality factor of the antenna in "free-space," i.e., it is independent of the propagation channel. Hence, in our analysis bandwidth considerations of electrically small antennas in fading channels remain the same as the ones predicted by Chu for "spherical antennas." Moreover, only dipole modes are used and link gain optimization based on the conjugate mode-matching criterion established in [20] applies. Hence, the optimum instantaneous effective gain is completely determined by the expansion coefficients of the propagation channel in spherical vector waves. This is a remarkable result since it means that for electrically small antennas, mode-matching of the lowest three modes (TE or TM) results in maximum link gain with channel adaptation while variable bandwidth is not required.

In the opposite limiting case when the size of the antenna becomes large compared to the wavelength, i.e., $ka \to \infty$, the antennas will show a potentially broadband behavior as predicted by Chu. In this case, when $Q_l < 1$, we set it to be unity since $Q_l < 1$ has no physical meaning in terms of bandwidth, i.e., the relative bandwidth cannot be larger than 200%. Therefore instead of the matrix containing the quality factor $\mathbf{Q}$, a modified matrix is used, $\mathbf{Q}^{+1} = \max[\mathbf{Q}, \mathbf{I}]$,[12] where $\mathbf{I}$ is the identity matrix. Hence, $Q \to 1$ as $ka \to \infty$ for all L. The distribution of the optimum G_i in this limit is independent of Q_l, for a constant L. Indeed, since Q_l is set to 1 as $ka \to \infty$ then[13]

$$G_\mathrm{i} = 4\pi \frac{\|\boldsymbol{a}_\tau\|_F^2}{\|\boldsymbol{a}\|_F^2}. \tag{23}$$

Hence, in the large frequency limit all modes up to L will contribute equally to the instantaneous effective gain.

In order to illustrate the theory derived above we now proceed to a numerical evaluation using a simple channel model (for justifications of the different assumptions in this model, see [20]) based on the more advanced channel models presented in [23], [24]. The model for the AoA for each of the two orthogonal polarizations assumes a two-dimensional Laplacian distribution in spherical coordinates, i.e., $p_{\theta,\phi\mathrm{x}}(\theta,\phi) = p_{\theta\mathrm{x}}(\theta)p_{\phi\mathrm{x}}(\phi) = A\exp(-\sqrt{2}|\theta-\mu_\theta|/\sigma_\theta - \sqrt{2}|\phi-\mu_\phi|/\sigma_\phi)\sin\theta$, where elevation angle $\theta \in [0,\pi]$, azimuth angle $\phi \in [0, 2\pi)$ and x stands for either of $\hat{\boldsymbol{\theta}}$- or $\hat{\boldsymbol{\phi}}$-polarization, and the shape is controlled by the distribution parameters $\{\mu_{\theta\mathrm{x}}, \sigma_{\theta\mathrm{x}}, \mu_{\phi\mathrm{x}}, \sigma_{\phi\mathrm{x}}\}$. We further assume that $\sigma = \sigma_{\theta\mathrm{x}} = \sigma_{\phi\mathrm{x}} = 0.1\ \mathrm{rad}$, $10\,\mathrm{rad}$ emulating channels of small and large angle spread, respectively; and $\mu_{\theta\mathrm{x}} = \pi/2$, $\mu_{\phi\mathrm{x}} = 0\ \mathrm{rad}$. Since (16)–(19) are independent of the cross-polarization ratio (XPR[14]) of the channel, 0 dB is used.

Fig. 1 illustrates the filtering effect of the radiation Q_l of the partial modes of the antenna on the transmission coefficients given in (16). The channel coefficients were obtained for a single realization of the channel according to the distribution described above for $\sigma = 0.1\ \mathrm{rad}$ and the maximum multi pole order $l = 2$. We clearly see that for a given electrical size ka of the antenna only a subset of all available modes will contribute to the radiated field due to the high losses associated with the higher order modes.

As we pointed out early, instantaneous parameters are based on quantities that can be modeled as continuous random variables, which are best described by probability distributions. Therefore, both G_i/Q, G_i and Q will be characterized by their cumulative distribution functions (cdf), [26], or rather by some values corresponding to some fixed probability levels, i.e., at some $p\%$ level of the cdf.

Fig. 2 shows the maximum G_i/Q at three probability levels, 1%, 50% and 99%, as a function of ka. Clearly, for small σ and constant L, the median ratio[15] as well as the spread[16] around the median increase with the electrical length ka. Observe that the median converges[17] to 2π as kr increases since we have only TM or TE modes at our disposal, but not both, therefore in average only half of the available power is used. However, for

[10]Even if deemed not practical a variable Q means that the bandwidth of the antenna must also change accordingly.

[11]The 0.5 threshold is used since for $kr \lesssim 0.5$ the radiation quality factor $Q \gtrsim 10$.

[12]The max operator acts elementwise.

[13]Observe that $\|\boldsymbol{a}\|_F^2 = \sum_{\tau=1}^{2}\sum_{l=1}^{L}\sum_{m=-l}^{l}|a_{\tau m l}|^2$ and that $\|\boldsymbol{a}_\tau\|_F^2 = \sum_{l=1}^{L}\sum_{m=-l}^{l}|a_{\tau m l}|^2$ where $\tau = 1$ or 2.

[14]The XPR of a channel is defined as the ratio of the power of the vertically polarized waves to the power of horizontally polarized waves, [18].

[15]The median is obtained as the value corresponding to the 50% level of the cdf.

[16]Here the spread is evaluated as the difference between values corresponding to the 1% and 99% levels of the cdf.

[17]This is of course an intuitive result, which will be investigated in the future.

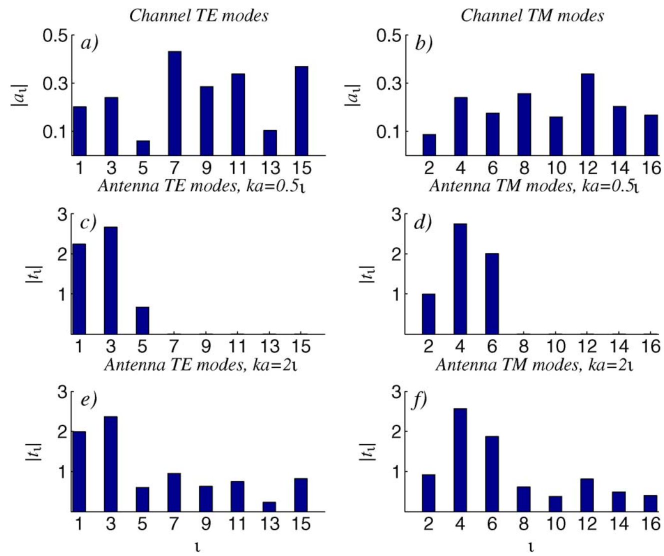

Fig. 1. Absolute value of the 8 lowest TE modes, a), and TM modes b), of a realization of the channel; antenna transmission coefficients obtained according to (16) for $ka = 0.5$ and TE modes, c) and TM modes, d), and antenna transmission coefficients for TE modes, e) and TM modes, f) for $ka = 2$.

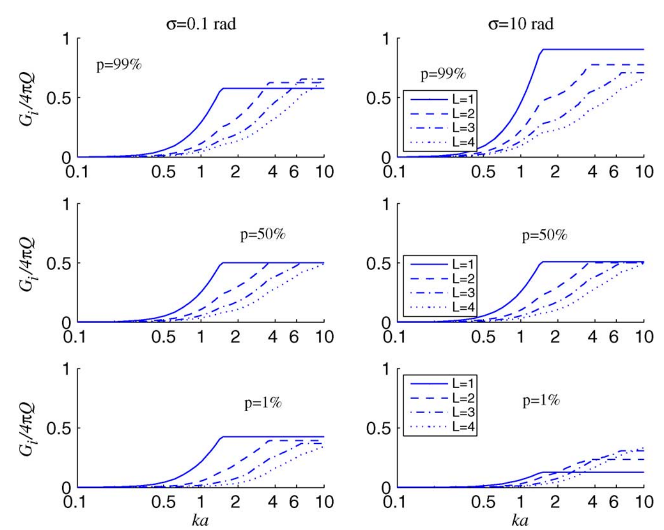

Fig. 2. Maximum G_{i}/Q in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained at three cdf levels: 1%, 50% and 99% and for $L = 1, 2, 3, 4$. When $Q_\iota < 1$, it is considered to be unity.

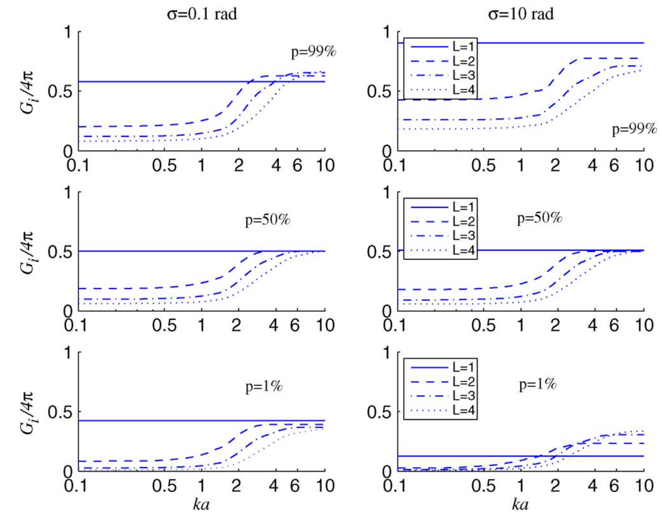

Fig. 3. G_{i} in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained at three cdf levels: 1%, 50% and 99% and for $L = 1, 2, 3, 4$. When $Q_\iota < 1$, it is considered to be unity.

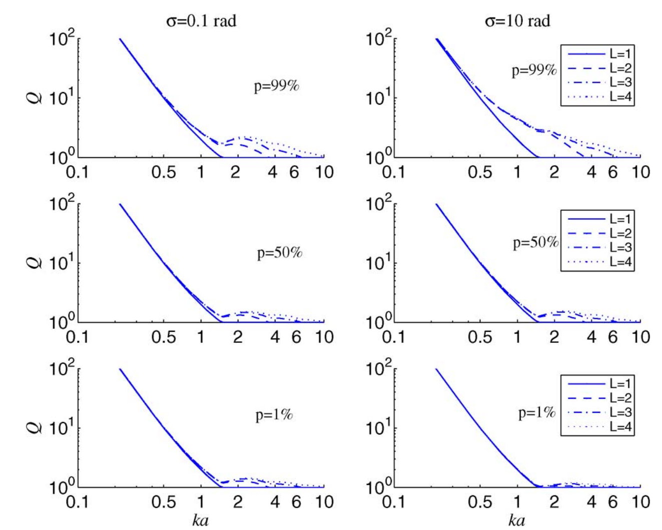

Fig. 4. Q in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained at three cdf levels: 1%, 50% and 99% and for $L = 1, 2, 3, 4$. When $Q_\iota < 1$, it is considered to be unity.

some realizations the mode matching can sometimes result in a better and sometimes worse G_{i}/Q depending on whether one of the TE or TM mode power is predominantly larger than the other for a given channel realization. For large σ the behavior of the median is similar to the small σ case. Namely, comparing the left and right plots we see that the small and large σ are basically identical. However, the spread around the median is much larger for the larger σ and it decreases with L. In the limiting case $ka \to \infty$ and for large L the variance converges to the same value independently of σ. This behavior can be better understood by examining the corresponding results for G_{i} and Q shown in Fig. 3 and Fig. 4, respectively. In the low ka limiting case G_{i} decreases as L is increasing since the performance of the antenna that excites modes with $L > 1$ "waste" their power due to large losses connected with large Q_l. As the electrical size of the antenna increases, exciting higher modes leads to an increase in the link power. However, this happens at the expense of smaller bandwidth as shown in Fig. 4. For a fixed L increasing ka results in an increase of G_{i} until a certain ka after which the cdf remains constant in the sense of the distribution parameters, with a well marked cutoff. Hence, no further mode "diversity gain" can be achieved due to limited degrees of freedom. Observe that G_{i} is independent of ka for $L = 1$. Clearly, to achieve better performance as ka increases, higher l-index multipoles should be excited and therefore L should be increased.

The smaller variance of G_{i} for small σ is explained by the fact that realizations of the channel modes, $\boldsymbol{a}$, are more correlated compared to that of large σ. However, as both L and ka increase,

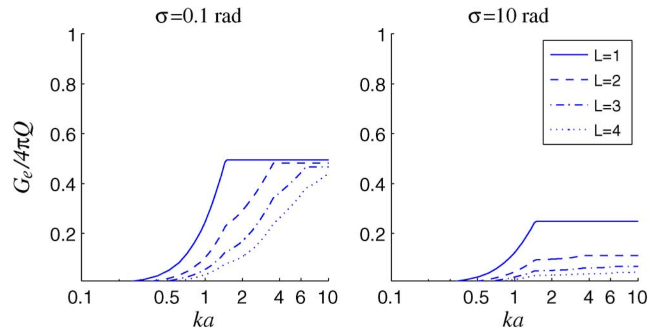

Fig. 5. Maximum G_e/Q in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained for $L = 1, 2, 3, 4$. When $Q_l < 1$, it is considered to be unity.

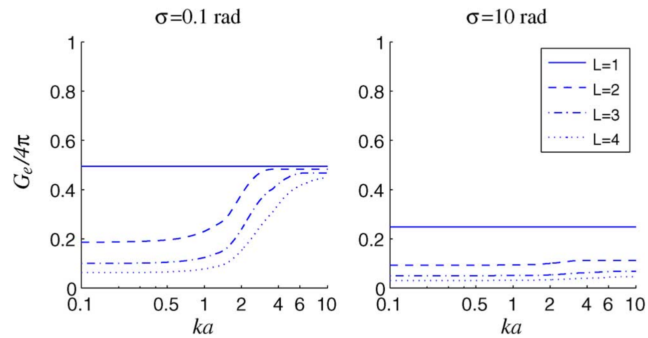

Fig. 6. G_{e} in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained for $L = 1, 2, 3, 4$. When $Q_l < 1$, it is considered to be unity.

the distribution of G_{i} seems to converge to the same function independently of σ. Certainly, increasing L for small σ and large ka implies an increase of degrees of freedom. However, since modes are correlated, the probability of actually "capturing" most of the power in the TM or TE modes with lower Q_l also increases, as it does increase the probability of capturing a lower share of the total mode power leading to the broadening effect of the distribution of G_{i} as ka increases for small σ. On the other hand for large σ channel modes are less correlated and therefore for small L the distribution of G_{i} is broader than for large L. Now, increasing L for large ka implies again an increase of degrees of freedom of the antenna resulting in averaging out the variation of G_{i}. The antenna radiation $Q \to 1$ as ka increases indicating that the antenna has a potentially broadband behavior.

B. Maximum G_{e}/Q of Ideal Antennas

The transmission coefficients that maximizes (13) with $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{R}}_{a_\tau}$ is given by the eigenvector, $\boldsymbol{t}_{\max}$, corresponding to the maximum eigenvalue, $\lambda_{\max}$, obtained solving the ordinary eigenvalue problem

$$\mathbf{Q}^{-1}\boldsymbol{\mathcal{R}}_{a_\tau}\boldsymbol{t} = \lambda\boldsymbol{t} \tag{24}$$

where $\boldsymbol{\mathcal{R}}_{a_\tau}$ is the correlation matrix of coefficients $\boldsymbol{a}_\tau$ corresponding to either TM or TE modes. Hence, values of G_{e}/Q, G_{e} and Q can be expressed as

$$\max\frac{G_{\mathrm{e}}}{Q} = 4\pi\frac{\lambda_{\max}}{\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}} \tag{25}$$

$$G_{\mathrm{e}} = \frac{\lambda_{\max}\boldsymbol{t}_{\max}^\dagger\mathbf{Q}\boldsymbol{t}_{\max}}{\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}} \tag{26}$$

$$Q = \frac{\boldsymbol{t}_{\max}^\dagger\mathbf{Q}\boldsymbol{t}_{\max}}{4\pi}. \tag{27}$$

Now for lossless, electrically small antennas we have again that only terms with $l = 1$ will be contributing to the interaction with the incident field. We have then for $ka \le 0.5$

$$Q \approx Q_1, \tag{28}$$

$$G_{\mathrm{e}} = 4\pi\frac{\lambda_{\max}^{(1)}}{\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}}, \tag{29}$$

$$\max\frac{G_{\mathrm{e}}}{Q} \approx \frac{4\pi}{Q_1}\frac{\lambda_{\max}^{(1)}}{\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}} \tag{30}$$

where $\lambda_{\max}^{(1)}$ is the maximum eigenvalue of the covariance matrix of the modes of the propagation channel $\boldsymbol{\mathcal{R}}_{\boldsymbol{a}_\tau}$, calculated for dipole modes only (TM or TE) for $L = 1$. These results are rather similar to the instantaneous channel results; however, they actually provide the performance limits when the correlation statistics are known rather than each particular realization of the channel. The maximum effective gain is obtained by exciting the strongest mode of the averaged channel.

In the high frequency limit, i.e., $ka \to \infty$, following the reasoning in the previous section $Q \to 1$, since $\mathbf{Q} \to \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Hence, the problem is reduced to obtaining the largest eigenvalue of the field correlation matrix containing TM or TE modes. The optimum MEG is then obtained as

$$G_{\mathrm{e}} = 4\pi\frac{\lambda_{\max}^{(L)}}{\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}} \tag{31}$$

where $\lambda_{\max}^{(L)}$ is the largest eigenvalue of the correlation matrix $\boldsymbol{\mathcal{R}}_{a_\tau}$.

Numerical simulations obtained with the simple channel model described above are shown in Figs. 4, 5 and 6. Fig. 5 shows the maximum G_{e}/Q as a function of ka for small and large σ and for different values of L. We see that for small σ and constant ka, G_{e}/Q decreases with increasing L due to the decrease of $\lambda_{\max}$ relative to the total available power $\mathrm{tr}\{\boldsymbol{\mathcal{R}}_a\}$. Now, fixing L but increasing ka results in larger G_{e} (Fig. 6) until a certain ka after which it remains constant with a well marked cutoff. Hence, no further mode "beamforming gain" can be achieved due to limited degrees of freedom. Observe that G_{e}/Q, $G_{\mathrm{e}} \le 2\pi$ if only TM or TE modes are used. At large σ the trend is somewhat different. Here, G_{e} is almost independent of ka for a constant L. On the other hand increasing L for constant ka will result in a decrease of G_{e} since the largest eigenmode $\lambda_{\max}^{(L)}$ will be much lower relative to the total power in a channels with large σ. Hence, using large antennas in spatially uniform channels is not optimal from the point of view of maximum link gain and dipole-like antennas are the most suitable, as expected. The bandwidth is only a concern for electrically small antennas and follows basically Q_1, while larger antennas again show potentially broadband behavior as shown in Fig. 7.

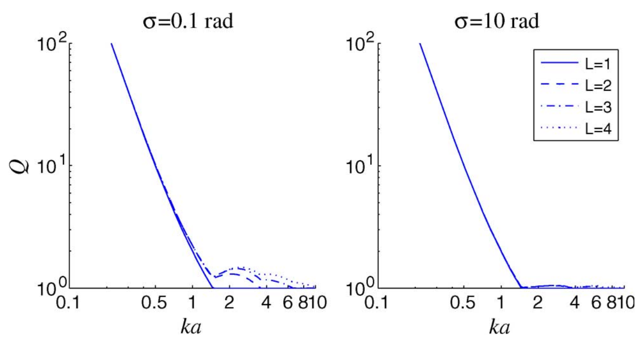

Fig. 7. Q in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained for $L = 1, 2, 3, 4$. When $Q_l < 1$, it is considered to be unity.

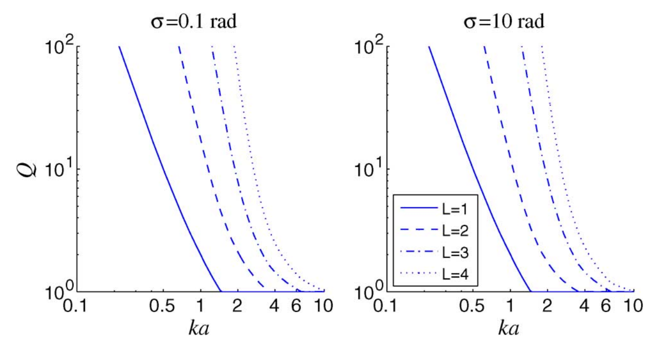

Fig. 9. Q corresponding to $G_{\mathrm{e}} = 4\pi$ in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained for $L = 1, 2, 3, 4$. When $Q_l < 1$, it is considered to be unity.

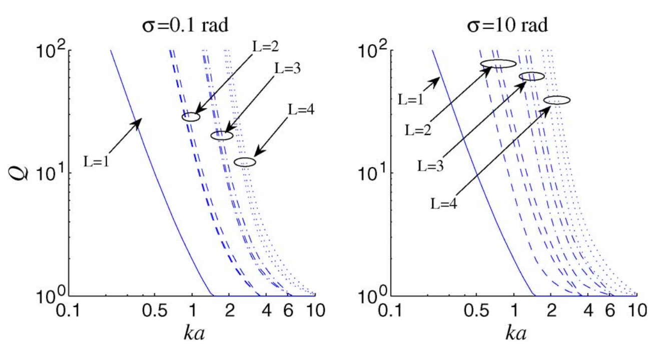

Fig. 8. Q corresponding to $G_{\mathrm{i}} = 4\pi$ in a propagation channel with small angle spread ($\sigma = 0.1$ rad) and in a propagation channel with large angle spread ($\sigma = 10$ rad). Results are obtained for $L = 1, 2, 3, 4$ denoted by $(-)$, $(--)$, $(.-)$ and $(..)$, respectively. Three cdf levels are considered, which are shown in a group of three curves for same L; the leftmost corresponds to 1%, the one in the middle to 50% and the rightmost to 99%. When $Q_l < 1$, it is considered to be unity.

C. *Q of ideal antennas for which $G_{\mathrm{i}} = 4\pi$ or $G_{\mathrm{e}} = 4\pi$*

Here we consider the radiation Q corresponding to optimal G_{i} and G_{e}. Now if we are interested in the "instantaneous" realizations of the channel, the optimal lossless antenna will have coefficients $\boldsymbol{t} = \boldsymbol{a}_\tau^*$ for an antenna exciting TM or TE modes only and $\boldsymbol{t} = \boldsymbol{a}^*$ for an antenna that excite both. In this case $G_{\mathrm{i}} = 4\pi$ independently of the channel and the highest order mode L used. The radiation Q will considerably increase in this case as shown in Fig. 8 for the TE- or TM-only case. Compare this result with Fig. 4. The utilization of both TM and TE modes will further increase the antenna Q but by a factor less than an order of magnitude. We see that in this case the bandwidth and therefore also the impedance of the antenna should be varied in order to keep maximum instantaneous gain.

In the case we just want to adapt to the covariance statistics of the channel we can maximize MEG, G_{e}, by transmitting with $\boldsymbol{t} = \boldsymbol{t}_{\max}$, where $\boldsymbol{t}_{\max}$ is the eigenvector corresponding to the maximum eigenvalue, $\lambda_{\max}$, of the mode covariance matrix $\boldsymbol{\mathcal{R}}_{a_\tau}$ or $\boldsymbol{\mathcal{R}}_a$ depending on whether only TE, TM or both modes are used, respectively. In this case also the link maximization will be achieved at the expenses of bandwidth reduction as depicted in Fig. 9; compare with Fig. 7.

IV. Conclusions

In this paper, we investigated physical limitations on the interactions of antennas exciting TM or TE modes with wireless propagation channels. The limitations are derived based on the spherical vector wave expansion of the electromagnetic field outside a sphere circumscribing the antennas. Rather than maximum antenna gain in a single direction we obtain physical limitations on the mean effective gain, which is a highly relevant quantity for multi-path propagation channels. The obtained limitations are on the maximum of G_{i}/Q and G_{e}/Q, which establish a trade-off between link gain and Q (and for narrowband antennas on bandwidth too). The main findings are summarized as follows: 1) if realizations of the channel are known, the transmission coefficients (or reception coefficients for reciprocal antennas) that optimize G_{i}/Q are given by the complex conjugate of the spherical vector wave expansion coefficients of the field impinging at the antenna, $\boldsymbol{a}_\tau$, weighted by the inverse of the radiation quality of the mode of order l, Q_l. Thus, the contribution of higher order modes will be attenuated as Q_l increases with the mode order l. 1a) The corresponding G_{i} and Q both depend on the realizations of $\boldsymbol{a}_\tau$ meaning that both the antenna gain pattern and the bandwidth of the antenna must changed adaptively. 1b) On the other hand for electrically small antennas the optimal bandwidth coincides with Chu's predictions and is independent of the channel, while the antenna gain pattern (dipole modes) must still be adaptively changed. 1c) Electrically small antennas are the most efficient with respect to the use of the available channel modes. 2) If the correlation matrix of the channel is known, the transmission coefficients (or reception coefficients for reciprocal antennas) that optimize G_{e}/Q are given by the eigenvector corresponding to the largest eigenvalue of the correlation matrix of the spherical vector wave expansion coefficients of the field impinging at the antenna, $\boldsymbol{\mathcal{R}}_{a_\tau}$, weighted by the inverse of the radiation quality of the mode of order l, Q_l. Here again, the contribution of higher order modes will be attenuated as Q_l increases with the mode order l. 2a) The corresponding G_{e} and Q both depend on the correlation matrix of $\boldsymbol{a}_\tau$ but are fixed when the statistics have been established. 2b) For electrically small antennas the optimal bandwidth coincides with Chu's predictions and is independent of the correlation properties of the channel, while the antenna gain pattern (dipole modes) still are.

2c) Electrically small antennas are the most efficient with respect to the use of the available channel modes in this case too. 3) The optimal performance of multi-port antenna systems with no mutual coupling, i.e., non-interacting ports is dictated by the optimal performance of the single-port antenna case since each port must have identical performance. These results are practically relevant especially for the design of antennas for cellular handsets, which operate in multipath environments, and require a good bandwidth as well as power efficiency.

APPENDIX A
SPHERICAL VECTOR WAVES

The regular spherical vector waves are given by

$$\mathbf{v}_{1ml}(k\boldsymbol{r}) = \mathrm{j}_l(kr)\mathbf{A}_{1ml}(\hat{\boldsymbol{r}}) \tag{32}$$

and

$$\mathbf{v}_{2ml}(k\boldsymbol{r}) = \frac{(kr\mathrm{j}_l(kr))'}{kr}\mathbf{A}_{2ml}(\hat{\boldsymbol{r}}) + \sqrt{l(l+1)}\frac{\mathrm{j}_l(kr)}{kr}\mathbf{A}_{3ml}(\hat{\boldsymbol{r}}) \tag{33}$$

where the time convention $e^{\mathrm{i}\omega t}$ is used, $\mathrm{j}_l(kr)$, $l = 1, 2, \ldots$ are the regular spherical Bessel functions, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, $r = |\boldsymbol{r}|$, and $|m| \leq l$.

Similarly, the incoming $(p = 1)$ and outgoing $(p = 2)$ spherical vector waves, $\mathbf{u}^{(p)}_{\tau ml}(k\boldsymbol{r})$ are given by

$$\mathbf{u}^{(p)}_{1ml}(k\boldsymbol{r}) = \mathrm{h}^{(p)}_l(kr)\mathbf{A}_{1ml}(\hat{\boldsymbol{r}}), \tag{34}$$

$$\mathbf{u}^{(p)}_{2ml}(k\boldsymbol{r}) = \frac{\left(kr\mathrm{h}^{(p)}_l(kr)\right)'}{kr}\mathbf{A}_{2ml}(\hat{\boldsymbol{r}}) + \sqrt{l(l+1)}\frac{\mathrm{h}^{(p)}_l(kr)}{kr}\mathbf{A}_{3ml}(\hat{\boldsymbol{r}}) \tag{35}$$

where $\mathrm{h}^{(p)}_l(kr)$ are the spherical Hankel functions of the p-th kind.

The functions $\mathbf{A}_{\tau ml}(\hat{\boldsymbol{r}})$ are the spherical vector harmonics that satisfy the complex valued inner product, i.e., orthogonality on the unit sphere [22]

$$\int \mathbf{A}_{\tau ml}(\hat{\boldsymbol{r}}) \cdot \mathbf{A}^*_{\tau' m' l'}(\hat{\boldsymbol{r}})\mathrm{d}\Omega = \delta_{\tau\tau'}\delta_{mm'}\delta_{ll'}. \tag{36}$$

APPENDIX B

Given $\mathbf{A} = \mathbf{A}^\dagger$ and $\mathbf{B}$ diagonal, the vector $\mathbf{x} \in \mathbb{C}$ that solves the optimization problem

$$\max \frac{\mathbf{x}^\dagger\mathbf{A}\mathbf{x}}{\mathbf{x}^\dagger\mathbf{B}\mathbf{x}} \tag{37}$$

is given by the eigenvector, $\mathbf{x}_{\max} \in \mathbb{C}$, that corresponds to the largest eigenvector $\lambda_{\max}$ of the matrix $\mathbf{B}^{-1}\mathbf{A}$, i.e.,

$$\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{x}. \tag{38}$$

Indeed, this is directly obtained from the Rayleigh-Ritz Theorem [27], applying variable substitution $\mathbf{y} = \mathbf{B}^{1/2}\mathbf{x}$. In the special case when $\mathbf{A} = \boldsymbol{a}\boldsymbol{a}^\dagger$ then the solution to (37) is given by

$$(\boldsymbol{a}^\dagger\mathbf{B}^{-2}\boldsymbol{a})^{-\frac{1}{2}}\mathbf{B}^{-1}\boldsymbol{a} = \mathbf{x}. \tag{39}$$

This can be seen[18] by substituting $\mathbf{A} = \boldsymbol{a}\boldsymbol{a}^\dagger$ into (38), which gives $\mathbf{B}^{-1}\boldsymbol{a}\boldsymbol{a}^\dagger\mathbf{x} = \lambda\mathbf{x}$. Then observing that $\boldsymbol{a}^\dagger\mathbf{x}$ is a constant. Hence, $\mathbf{x} = c\mathbf{B}^{-1}\boldsymbol{a}$, where c is a constant obtained from the normalization $\mathbf{x}^\dagger\mathbf{x} = 1$.

APPENDIX C

Given $\mathbf{A} = \mathbf{A}^\dagger$ and $\mathbf{B}$ diagonal, the matrix $\mathbf{X} \in \mathbb{C}$ that solves the optimization problem

$$\max \frac{\mathrm{tr}\{\mathbf{X}\mathbf{A}\mathbf{X}^\dagger\}}{\mathrm{tr}\{\mathbf{X}\mathbf{B}\mathbf{X}^\dagger\}} \tag{40}$$

is given by the matrix $\mathbf{X}_{\max}$ containing N identical row-vectors each equal $\boldsymbol{x}^\dagger_{\max}$ where $\boldsymbol{x}_{\max}$ is given by the solution to (37). To show this result we apply $\partial/\partial\mathbf{X} = 0$ to (40), which gives

$$\frac{\mathbf{X}^*\mathbf{A}^t\mathrm{tr}\{\mathbf{X}\mathbf{B}\mathbf{X}^\dagger\} - \mathrm{tr}\{\mathbf{X}\mathbf{A}\mathbf{X}^\dagger\}\mathbf{X}^*\mathbf{B}^t}{\left(\mathrm{tr}\{\mathbf{X}\mathbf{B}\mathbf{X}^\dagger\}\right)^2} = 0 \tag{41}$$

where we used the identity $\partial\mathrm{tr}(\mathbf{X}\mathbf{A}\mathbf{X}^\dagger)/\partial\mathbf{X} = \mathbf{X}^*\mathbf{A}^t$. Observing that expressions of the form $\mathrm{tr}\{\mathbf{X}\mathbf{B}\mathbf{X}^\dagger\}$ are scalars and can be considered a constant for a given $\mathbf{X}$ and that $\mathbf{B}^{-1}$ exists yields

$$\mathbf{B}^{-1}\mathbf{A}\mathbf{X}^\dagger - \lambda\mathbf{X}^\dagger = 0. \tag{42}$$

Now using the identities $\mathrm{vec}(\mathbf{A}\mathbf{B}\mathbf{C}) = (\mathbf{C}^t \otimes \mathbf{A})\mathrm{vec}(\mathbf{B})$ and $(\mathbf{A} \otimes \mathbf{C})(\mathbf{B} \otimes \mathbf{D}) = (\mathbf{A}\mathbf{B} \otimes \mathbf{C}\mathbf{D})$ we arrive at the eigenvalue problem formulation

$$\left(\mathbf{I} \otimes \mathbf{B}^{-1}\mathbf{A}\right)\mathrm{vec}(\mathbf{X}^\dagger) = \lambda\mathrm{vec}(\mathbf{X}^\dagger) \tag{43}$$

which can be obtained by solving

$$\det\left(\mathbf{I} \otimes (\mathbf{B}^{-1}\mathbf{A} - \lambda\mathbf{I})\right) = 0. \tag{44}$$

Further, observing that $\det(\mathbf{A} \otimes \mathbf{B}) = \det(\mathbf{A})^{\mathrm{rank}(\mathbf{B})}\det(\mathbf{B})^{\mathrm{rank}(\mathbf{A})}$ and since $\mathrm{rank}(\mathbf{I}) = N$ and $\det(\mathbf{I}) = 1$, then (44) is reduced to N identical equations

$$\det\left(\mathbf{B}^{-1}\mathbf{A} - \lambda\mathbf{I}\right)^{\mathrm{N}} = 0. \tag{45}$$

Hence, the solution to (43) is then given by the matrix $\mathbf{X}_{\max}$ containing N identical raw-vectors each equal $\boldsymbol{x}^\dagger_{\max}$ where $\boldsymbol{x}_{\max}$ is the solution corresponding to the maximum eigenvalue (38). In the special case when $\mathbf{A} = \boldsymbol{a}\boldsymbol{a}^\dagger$ then the solution to (40) is given by

$$\mathbf{X}_{\max} = [\boldsymbol{x}_{\max} \quad \cdots \quad \boldsymbol{x}_{\max}]^\dagger \tag{46}$$

where $\boldsymbol{x}_{\max} = (\boldsymbol{a}^\dagger\mathbf{B}^{-2}\boldsymbol{a})^{-(1/2)}\mathbf{B}^{-1}\boldsymbol{a}$.

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[18]Alternatively, observe that $\mathrm{rank}(\mathbf{B}^{-1}\mathrm{a}\mathrm{a}^\dagger) = 1$ and therefore $\det(\mathbf{B}^{-1}\boldsymbol{a}\boldsymbol{a}^\dagger - \lambda\mathbf{I}) = 0$ has a single non-zero eigenvalue.

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